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A MODIFICATION OF THE WILSON & JUDGE DEPOSIT VELOCITY EQUATION, EXTENDING ITS APPLICABILITY TO FINER PARTICLES AND LARGER PIPE SIZES

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Abstract: This paper is concerned with predicting the deposit velocity of turbulently flowing, fine particle slurries in a horizontal pipe. The Wilson and Judge (1976) deposit velocity correlation is widely used to predict the deposit velocity in turbulent pipe flow of sand in water slurries. However the W&J correlation is limited to medium size particles as the pipe size increases. For example, for silica sand in water in a 1000 mm pipe, the W&J correlation only predicts V_d adequately for d_{50} particle sizes 150 µm and coarser. Thomas (2014) showed how for most wide size distribution, viscous slurries pumped in the mining industry, the equivalent sand in water particle size is less than 100 µm. This means that the W&J correlation is generally of limited applicability for these slurries, especially in large pipe sizes. A Modified Wilson & Judge (MW&J) correlation is presented. Predicted deposit velocities using MW&J follow smooth curves up to 1000 mm pipe diameter for silica sand particles in water down to 30 µm. Some experimental results from the literature for sands in viscous fluids are compared with predicted deposit velocities using the modified equation. The relevance of the modified equation to deposit velocity predictions for the typical wide size distribution, viscous concentrates and tailings slurries pumped in the mining industry, is discussed.

1. INTRODUCTION

The deposit velocity (V_d) is defined as the velocity at which a stationary bed of solids first appears as the velocity is progressively reduced. The Wilson and Judge (W&J) (1976) deposit velocity correlation involves a parameter Δ which is a function of particle settling velocity, pipe diameter and solids SG. The W&J correlation applies to $\Delta > 10^{-5}$. The 10^{-5} minimum on Δ limits the applicability of the W&J correlation to medium size particles as the pipe size increases. For sand in water slurries this particle size limit may not seem too restrictive but it severely affects the applicability of the W&J correlation for most slurries pumped in the mining industry. Thomas (2014) showed how the d_{50} particle size of these wide size distribution slurries is generally less than 100 µm. Because these slurries possess a viscosity higher than the viscosity of water, the particle settling velocity of a 100 µm d_{50} size particle is equivalent to a much finer sand particle in water. For example the d_{50} = 100 µm particle could be equivalent to a 50 µm or 60 µm sand particle in water. The W&J correlation is therefore of limited use for the majority of wide size distribution slurries pumped in the mining industry.

2. CURRENT DEPOSIT VELOCITY PREDICTION METHODS FOR FINE PARTICLES

2.1 WILSON AND JUDGE (1976)

The W&J (1976) deposit velocity correlation was the basis for the deposit velocity predictions for medium size sands in the nomograph presented by Wilson and Judge (1978). The deposit velocity, V_d , is the maximum deposit velocity for the particular particle size, solids SG, and pipe diameter. This maximum deposit velocity will relate to a particular concentration and V_d for other concentrations is always less than that given by the nomograph, although the dependence on concentration for fine particle slurries is small, (see Section 3). V_d is given in terms of the familiar Durand (1953) equation, where g is the gravitational constant, D is pipe diameter (m), and S is the ratio of density of particles to density of fluid (ρ_p/ρ_f). The W&J F_L parameter is as given below where W is the settling velocity of particle in quiescent fluid (m/s) and d is particle diameter (m).

$$V_d = F_L\,[2gD(S-1)]^{0.5} \tag{1}$$

$$F_L = 2.0 + 0.3\,\log_{10}\Delta \tag{2}$$

$$\Delta = 0.75\,W^2/[g\,D\,(S-1)] = d/(D\,C_d) \tag{3}$$

$$C_d = \text{particle drag coefficient} = 4gd(S-1)/(3W^2) \tag{4}$$

The particle settling velocity of a granular particle is assumed as for a sphere. Schriek et al (1973) conducted settling tests on a range of individual sand particles and found that the particle drag coefficients were essentially the same as for a sphere. The mean of each mesh size range represented the characteristic particle size. In a quiescent fluid a particle will align itself so as to result in the highest drag and therefore lowest settling velocity. However in a turbulent flow field the particle will present all alignments to the rapidly changing velocity field, thereby effectively giving an average equivalent settling velocity even closer to that of a sphere.

2.2 VISCOUS SUB-LAYER DEPOSITION – THOMAS (1979)

Thomas (1979) developed a deposit velocity prediction method for particles smaller than the viscous sub-layer, based on the Wilson sliding bed theory. Thomas argued that Equation 5 provides a lower limit to the deposit velocity.

$$V_{d\delta}^{*} = 1.1\,[g\,\mu_f(\rho_p - \rho_f)/\rho_f^2\,]^{1/3} \tag{5}$$

The dδ subscript identifies it as the viscous sub-layer deposit velocity. $V_{d\delta}^*$ is friction velocity at deposition = $V_{d\delta}\sqrt{(f/2)}$, (m/s), f = Fanning friction factor at the deposit velocity, and μ_f = viscosity of fluid (Pa.s). Thomas (1979) defined the viscous sub-layer thickness, δ, as:

$$\delta = 5\ \mu_f /(\rho_f\ V^*) \tag{6}$$

where V^* is friction velocity = $V\sqrt{(f/2)}$, and f is friction factor at velocity V. Therefore, according to Eqn 5, the thickness of the viscous sub-layer at deposition conditions (δ_d) is given by:

$$\delta_d = 5\ \mu_f /(\rho_f\ V_{d\delta}^*) \tag{7}$$

2.3 SANDERS ET AL (2004)

Sanders et al (2004) modified Thomas' (1979) viscous sub-layer theory to include the effect of particle size and concentration. Their deposit velocity, given in terms of the friction velocity at deposition, (termed here V_{dSa}^*) is as follows:

$$1.1\ V_{dSa}^*/ V_{d\delta}^* = [\ 0.76 + 0.15\ d\rho_f\ V_{dSa}^*/\mu_f\ ] / [(C_{max} - C_v)^{0.88}]^{1/3} \tag{8}$$

C_{max} is the maximum packing concentration by volume and C_v is the volume concentration of interest. $V_{d\delta}^*$ is the viscous sub-layer friction deposit velocity of Thomas (1979) given by Eqn 5. Equation 8 applies to d^+ (= $d\ \rho_f\ V_{dSa}^*/ \mu_f$) < 5.

3. NEW MODIFIED WILSON & JUDGE CORRELATION

Using an informed trial and error approach, and after trying many different variations, a modification to Equation 2 was arrived at. The new, Modified Wilson & Judge (MW&J) correlation for F_L is:

$$F_L = 2 + 0.305 \log_{10}\Delta + 1.1\text{x}\ 10^{-4}\ \Delta^{-0.489} - 0.044\ (1\text{x}10^7\ \Delta)^{-1.06} \tag{9}$$

In considering Equation 9, it should be noted that Wilson & Judge's equation for F_L (Eqn 2) was not a direct theoretically derived relationship but represented a best fit correlation to the results from their individual computer analyses for various combinations of particle size and pipe size. With this consideration taken into account the form of Equation 9 would seem equally valid. F_L from Eqn 9 is inserted into Eqn 1 to give the predicted MW&J deposit velocity, $V_{d\Delta}$. The dΔ subscript identifies it as the deposit velocity predicted using MW&J based on Δ.

Figure 1 compares MW&J $V_{d\Delta}$ predictions for 200 μm, 150 μm, 100 μm, 75 μm and 55 μm sand in water, with the W&J predictions (Eqn 2). The thick grey lines are the W&J predictions and the thin black lines are the MW&J predictions. The W&J predictions for 200 μm and 150 μm particles are shown as being applicable up to the 1000 mm pipe diameter although it is likely that W&J (and hence MW&J) under-predict for particles greater than 150 µm in a 1000 mm pipe. The $\Delta<10^{-5}$ criterion limits the applicable W&J pipe size for particles less than 150 µm as indicated, i.e. maximum 325 mm pipe size for 150 µm particle size, maximum 115 mm pipe size for 75 µm particle size, and maximum 35 mm pipe size for 55 µm particle size. The thick black curve near the bottom of the graph is the viscous sub-layer prediction, $V_{d\delta}$ of Thomas (1979), Eqn 5.

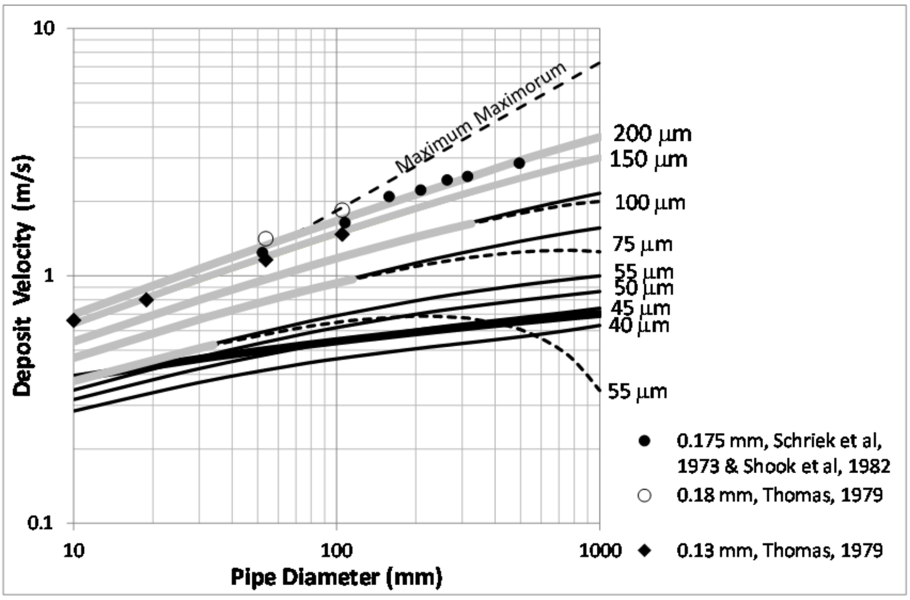

Figure 1 Deposit velocity vs pipe diameter – Comparisons MW&J and W&J predictions

The MW&J predictions are similar (within 1%) to the original W&J predictions up to the original limit of $\Delta = 10^{-5}$. The dashed curves are the extensions to the original W&J predictions for $\Delta < 10^{-5}$ and show increasing deviation from the obvious behaviour trends as the particle size decreases and pipe size increases. In a 1000 mm pipe this deviation is significant for particle sizes less than 100 µm. Also shown in Figure 1 are MW&J $V_{d\Delta}$ predictions for 50 µm, 45 µm and 40 µm particles. For industrially important pipe sizes from 100 mm to 1000 mm, the predicted $V_{d\Delta}$ for 45 µm particle size is similar to the viscous sub-layer (Eqn 5) prediction. For 40 µm particle size $V_{d\Delta}$ is approximately 0.1 m/s below $V_{d\delta}$. Since, for 45 µm particle size, $V_d = V_{d\delta}$, therefore for particles smaller than 45 µm, $V_d = V_{d\delta}$ since $V_{d\delta}$ represents a lower limit to the deposit velocity according to Thomas (1979).

Also shown in Figure 1 are observed deposit velocities (all for Cv = 0.12) for 0.18 mm sand in 53.8 mm and 105 mm, 0.175 mm sand in pipe sizes from 52.2 mm to 495 mm, and 0.13 mm sand in pipe sizes from 9.41 mm to 105 mm. These data points are consistent with the MW&J predictions for these particle sizes and follow the predicted slope. Of course these data also follow the original W&J prediction. Note that the predicted slope and slope of the data, is significantly less than the slope of the Maximum Maximorum line of Wilson applying to coarse particles (see Wilson et al, 2010).

So far no mention has been made of the effect of concentration on the deposit velocity. In the author's experience (e.g. Thomas, 1979), the maximum V_d for medium to fine sand in water slurries almost always occurs for volume concentrations around 0.12. The wide ranging data of Schriek et al (1973) show a similar result. The two sets of results indicate that for volume concentrations in the range 0.12 to 0.38, V_d is generally within 10% of the value at Cv = 0.12. Hence the MW&J predictions for sand-water slurries in this paper are assumed to apply for concentrations in this range. The exact effect of concentration on MW&J predictions for more viscous slurries has not yet been investigated.

In Figure 1 the predictions of MW&J and W&J were plotted against pipe diameter. The predictions of MW&J with W&J can also be plotted against particle size for a particular pipe diameter. Figure 2 shows such a plot for silica sand in water for pipe diameters 1000 mm, 325 mm and 115 mm. (Solids density 2650 kg/m^3, fluid density 1000 kg/m^3, Cv = 0.12, viscosity 1 mPas, pipe roughness 0.01 mm). The thick grey curves indicate the predictions of W&J down to the particle size for which $\Delta = 10^{-5}$, with the thin line extensions for $\Delta < 10^{-5}$ representing the MW&J predictions. Also shown as horizontal lines in Figure 2, are the viscous sub-layer ($V_{d\delta}$) predictions from Equation 5, for the three pipe sizes, ranging from 0.70 m/s in the 1000 mm pipe to 0.55 m/s in the 115 mm pipe, as well as the $V_{d\delta}$ prediction for a 18.9 mm pipe (0.43 m/s). For convenience these are shown extending across to 160 μm particle size although once $V_{d\Delta}$ exceeds $V_{d\delta}$, the latter no longer applies. Observed deposit velocities in an 18.9 mm pipe for 17 μm and 26 μm sand, (Thomas, 1979), agree well with the $V_{d\delta}$ prediction as shown by the two open circle data points.

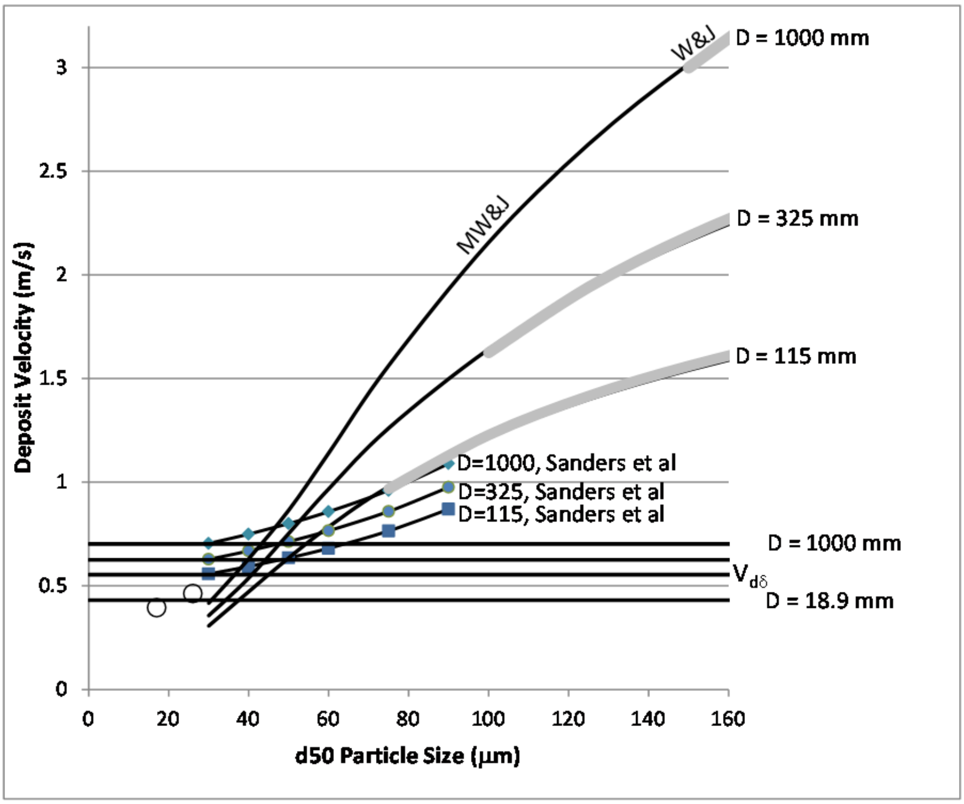

Figure 2 MW&J Predictions for Sand in Water c.f. Sanders et al (2004) and Thomas (1979)

Also shown in Figure 2 are predictions using the method of Sanders et al (2004), Eqn 8, for 1000 mm, 325 mm and 115 mm pipes, shown extending to the $d^+ = 5$ limit, as recommended by those authors. The Sanders et al predictions provide a smooth increase above $V_{d\delta}$ as the particle size increases but once they cross the MW&J curves the predicted trends are at variance with MW&J predicted trends. This discrepancy between the two models requires further investigation and comparison with appropriate data when it becomes available. It could be that the Sanders et al predictions may only apply until they cross the the MW&J curves, in which case, for the predictions shown ($C_v = 0.12$) they would apply for particle sizes from about 30 μm to 50 μm. For concentrations greater than 0.12, the Sanders et al predicted curves rise more steeply and will cut the MW&J curves at a particle size greater than the 50 μm indicated in Figure 2.

The MW&J predictions cut the $V_{d\delta}$ predictions at 43 μm, 44 μm, and 45 μm respectively for the 1000 mm, 325 mm and 115 mm pipe diameters. These critical particle sizes are in agreement with the observation in regard to Figure 1, that $V_{d\Delta}$ is similar to $V_{d\delta}$ at approximately 45 μm particle size.

From Eqn 7, for the particular case of silica sand in 20^0C water, at $V_{d\delta}$ the viscous sub-layer thickness δ_d = 180 μm. Figure 2 shows that $V_{d\Delta} = V_{d\delta}$ for a particle size around 45 μm for industrially relevant pipe sizes from 100 mm to 1000 mm. Hence at viscous sub-layer deposition, the ratio $d/\delta = 45/180 = 0.25$. Interestingly, it appears that according to the MW&J correlation, viscous sub-layer deposition becomes the controlling criterion when d/δ becomes less than 0.25, for a wide ranging combination of particle size, solids density, fluid viscosity and fluid density. For example Table 1 summarises MW&J predictions for four combinations of solids density, fluid density and fluid viscosity. For each case the particle size has been adjusted until the predicted deposit velocity using the MW&J correlation ($V_{d\Delta}$) equals the predicted viscous sub-layer deposit velocity ($V_{d\delta}$) for industrially relevant pipe sizes from 100 mm to 1000 mm. The particle size at which this occurs is shown in Table 1 together with the viscous sub-layer thickness calculated as per Equation 7. In each case the ratio of particle size to viscous sub-layer thickness (d/δ_d) equals 0.25. Thus, based on the predictions of MW&J, viscous sub-layer deposition becomes the controlling criterion when d/δ becomes less than 0.25. This is in reasonable agreement with the data of Thomas (1979) which indicated deposition was controlled by viscous sub-layer deposition for $d/\delta < 0.3$.

Table 1

Solids Density (kg/m^3)	Fluid Density (kg/m^3)	Fluid Viscosity (mPas)	Particle Size for $V_{d\Delta} = V_{d\delta}$ (μm)	Viscous Sub-layer Thickness, δ_d (μm)	d/δ_d
2650	1000	1	45	180	0.25
2650	1200	3	93	367	0.25
5000	1000	1	33	134	0.25
1400	1050	2	120	471	0.25

4. RELEVANCE TO SLURRIES IN THE MINING INDUSTRY

The MW&J prediction method is aimed at predicting the deposit velocity for fine particle sizes equivalent to sand in water particles 100 μm to 45 μm so as to be relevant to wide size distribution, viscous slurries of interest to the mining industry. Unfortunately the author has been unable to find any sand-water data for which $\Delta < 1.E\text{-}5$ and $V_{d\Delta} > V_{d\delta}$. Such data is more likely to apply to large pipe diameters so is understandably scarce.

However there is some limited data for sand in fine particle carrier fluids. Table 2 summarises relevant data from Sanders et al (2004), with their Figure 5 being relevant to Slurries 1 and 2, and their Figure 10 being relevant to Slurry 3. Additional information regarding Slurry 3 was provided by Sanders (2015). For Slurries 1 and 2 the observed V_d are not much higher than $V_{d\delta}$ predicted by Thomas (1979), Eqn 5, namely 0.43 m/s and 0.23 m/s higher respectively, and the predicted V_d by MW&J and Sanders et al could be considered equally relevant. For slurry 3 the observed V_d range is 0.78 m/s to 0.98 m/s above $V_{d\delta}$. The MW&J predicted 1.82 m/s is 0.22 m/s higher than the mean observed V_d whilst the 1.35 m/s predicted by Sanders et al is 0.25 m/s less than the mean observed V_d. So once again MW&J and Sanders et al could be considered equally relevant. Further resolution of these comparisons will require more data in the relevant range for which $\Delta < 1.E\text{-}5$ and $V_{d\Delta} > V_{d\delta}$.

Table 2 Data for which $\Delta < 10^{-5}$ and $V_{d\Delta} > V_{d\delta}$

No.	D mm	d_{50} µm	ρ_f kg/m^3	μ_f mPas	Δ	Observed V_d m/s	Predicted V_d MW&J m/s	Predicted V_d Sanders et al m/s
1	264	169	1145	3.5	9.7E-6	1.18	1.31	1.02
2	264	169	1170	6	3.4E-6	1.07	0.98	0.98
3	495	158	1123	2.12	9.7E-6	1.5-1.7	1.82	1.35

Thomas (2014) considered the properties of 39 concentrates and 105 tailings slurries at 60% concentration and analysed the data using the inherent viscosity approach of Thomas (2010). For the concentrates he found that the equivalent sand in water particle size was 23 µm for the average size concentrate and 113 µm for the coarsest concentrate. Figure 2, applicable to sand in water, indicates that for particles less than about 45 µm, V_d is given by $V_{d\delta}$. It could be concluded that V_d for the average size concentrate would similarly be given by $V_{d\delta}$ and V_d for the coarsest concentrate would be given by MW&J, $V_{d\Delta}$. Actual predictions of V_d for concentrates in a typical 300 mm diameter pipe confirmed these conclusions. The cross over occurs at a weighted mean particle size (d_m) $\cong$ 50 µm. For the average concentrate, the predicted $V_d = V_{d\delta} = 1.1$ m/s and for the coarsest concentrate, $V_d = V_{d\Delta} = 1.75$ m/s. These predictions do not take into account laminar/turbulent transition, which depends on slurry yield stress, and will often control deposition.

For the tailings, Thomas (2014) found that the equivalent sand in water particle size was 22 µm for the average size tails and 128 µm for the coarsest tails. Prediction of V_d for tailings in a typical 600 mm diameter pipe, confirmed that V_d for the average tails is determined by $V_{d\delta}$, and for the coarsest tails by $V_{d\Delta}$, i.e. by MW&J. The cross over occurs for $d_m \cong 125$ µm. For the average tails, the predicted $V_d = 1.35$ m/s and for the coarsest, $V_d = 2.2$ m/s. As noted above, all these calculations assume 60% concentration in all cases. For concentrations less than 60% $V_{d\Delta}$ will become more relevant than $V_{d\delta}$.

5. CONCLUSIONS

The MW&J predicted curves of $V_{d\Delta}$ versus D in Figure 1 for sand in water, have the following properties:

- The predictions are within 1% of the Wilson & Judge (1976) predictions for $\Delta > 10^{-5}$ for the range of D relevant to each particle size.
- The MW&J predictions appear to be reasonable extrapolations of the relevant W&J predictions.
- As the particle size decreases, the predicted $V_{d\Delta}$ approaches the same slope as the viscous sub-layer prediction, $V_{d\delta}$, of Thomas (1979), at least for the industrially meaningful pipe size range of 100 mm to 1000 mm.
- The MW&J predicted deposit velocity, $V_{d\Delta}$, equals, and then becomes less than $V_{d\delta}$, at a particles size around 45 μm.

Thus the MW&J correlation extends the W&J correlation to provide a good connection between the approaches of Wilson and Judge (1976) and Thomas (1979).

On a plot of V_d versus particle size (Figure 2), the MW&J predicted curves cross the viscous sub-layer predictions at about 45 μm equating to $d/\delta = 0.25$. Table 1 shows that the same value of d/δ at cross-over applies for a range of solid and fluid densities and fluid viscosities. This indicates that viscous sub-layer deposition becomes the controlling criterion when d/δ becomes less than 0.25.

Figure 2 also compares predictions of Sanders et al (2004) with MW&J. For particle sizes from about 30 μm to 50 μm the Sanders et al correlation could be considered to be providing a transition between $V_{d\delta}$ and $V_{d\Delta}$. The Sanders et al predictions in Figure 2 apply to $C_v = 0.12$. The Sanders et al correlation includes concentration as a parameter and for concentrations greater than 0.12, the Sanders et al predicted curves rise more steeply and will cut the MW&J curves at a particle size greater than the 50 μm indicated in Figure 2.

Finally it must be emphasised that the MW&J correlation has not been tested against data from large pipes and therefore caution must be exercised when making predictions in very large pipes. Also the MW&J correlation does not include coarse solids concentration as a parameter. For medium to fine sands in water, experimental data suggest that the MW&J predicted V_d will be within 10% of the maximum V_d in the practical concentration range. For more viscous slurries this may not be the case.

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